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Answers and Explanations to AB Practice Exam 2

ANSWERS AND EXPLANATIONS TO SECTION I

Problem 1. If $g(x)=\frac{1}{32}x^4-5x^2$, find $g'(4)$.

First, take the derivative.

$$g'(x)=\frac{1}{32}\left(4x^3\right)-5(2x)=\frac{x^3}{8}-10x$$

Now, Plug In 4 for x.

$$\frac{(4)^3}{8}-10(4)=8-40=-32$$

The answer is (B).

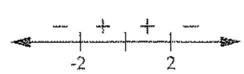
Problem 2. The domain of the function $f(x)=\sqrt{4-x^2}$ is

When you have a square root in a function, the domain will require that the expression under the radical (the "radicand") not be negative. Thus, the domain will be those values where $4-x^2$ is not negative.

In other words, $4-x^2\geq 0$.

We solve this by, first, factoring the expression on the left. $(2+x)(2-x)\geq 0$

Next, we take the roots of the left side, which are –2 and 2, and put them on a number line.



Now, we pick a value in each of the three regions on the number line $x<-2$, $-2<x<2$, and $x>2$. We plug the value into the expression $4-x^2$ to see if we get a positive or negative value. If it's positive, then we include that region in the domain. If it's negative, then we exclude that region from the domain.

Let's try -3 for a value in the region $x<-2$.

We get: $4-(-3)^2=-5$, so we exclude the region $x<-2$ from the domain.

Now, we try 0 for a value in the region $-2<x<2$.

We get: $4-(0)^2=4$, so we include the region $-2<x<2$ in the domain.

Finally, we try 3 for a value in the region $x>2$.

We get: $4-(3)^2=-5$, so we exclude the region $x>2$ from the domain.

Because the radicand is allowed to be zero, we include the endpoints in the domain. Therefore, the domain is $-2\leq x\leq 2$.

The answer is (D).

Problem 3. $\lim_{x \to 5} \dfrac{x^2 - 25}{x - 5}$ is

Notice that if we plug 5 into the expressions in the numerator and the denominator, we get: $\dfrac{0}{0}$, which is undefined. Before we give up, we need to see if we can simplify the limit so that it can be evaluated. If we factor the expression in the numerator, we get: $\dfrac{(x+5)(x-5)}{(x-5)}$, which can be simplified to $x+5$.

Now, if we take the limit (by Plugging In 5 for x), we get 10.

The answer is (B).

Problem 4. If $f(x) = \dfrac{x^5 - x + 2}{x^3 + 7}$, find $f'(x)$.

We need to use the Quotient Rule, which is:

Given $f(x) = \dfrac{g(x)}{h(x)}$ then $f'(x) = \dfrac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$

Here, we have:

$$f'(x) = \frac{(x^3+7)(5x^4-1) - (x^5 - x + 2)(3x^2)}{(x^3+7)^2}$$

The answer is (E).

Problem 5. Evaluate $\lim_{h \to 0} \dfrac{5\left(\frac{1}{2} + h\right)^4 - 5\left(\frac{1}{2}\right)^4}{h}$.

Notice how this limit takes the form of the definition of the Derivative, which is:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Here, if we think of $f(x)$ as $5x^4$, then this expression gives the derivative of $5x^4$ at the point $x = \dfrac{1}{2}$.

The derivative of $5x^4$ is $f'(x) = 20x^3$.

At $x = \dfrac{1}{2}$, we get $f'\left(\dfrac{1}{2}\right) = 20\left(\dfrac{1}{2}\right)^3 = \dfrac{5}{2}$

The answer is (A).

PROBLEM 6. $\int x\sqrt{3x}\,dx =$

First, rewrite the integral as: $\int x \cdot \sqrt{3} \cdot x^{\frac{1}{2}}\,dx$.

Now we can simplify the integral to: $\sqrt{3}\int x^{\frac{3}{2}}\,dx$.

Using the power rule for integrals, which is $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$,

we get: $\sqrt{3}\int x^{\frac{3}{2}}\,dx = \sqrt{3}\frac{x^{\frac{5}{2}}}{\frac{5}{2}} + C = \frac{2\sqrt{3}}{5}x^{\frac{5}{2}} + C$

The answer is (A).

PROBLEM 7. Find k so that $f(x) = \begin{cases} \frac{x^2-16}{x-4} & ; x \neq 4 \\ k & ; x = 4 \end{cases}$ is continuous for all x.

In order for $f(x)$ to be continuous at a point c, there are three conditions that need to be fulfilled:

(1) $f(c)$ exists.

(2) $\lim_{x\to c} f(x)$ exists.

(3) $\lim_{x\to c} f(x) = f(c)$.

First, let's check condition (1). $f(4)$ exists; it's equal to k.

Next, let's check condition (2). From the left side, we get:

$$\lim_{x\to 4^-}\frac{x^2-16}{x-4} = \lim_{x\to 4^-}\frac{(x-4)(x+4)}{x-4} = \lim_{x\to 4^-}(x+4) = 8$$

From the right side, we get:

$$\lim_{x\to 4^+}\frac{x^2-16}{x-4} = \lim_{x\to 4^+}\frac{(x-4)(x+4)}{x-4} = \lim_{x\to 4^+}(x+4) = 8$$

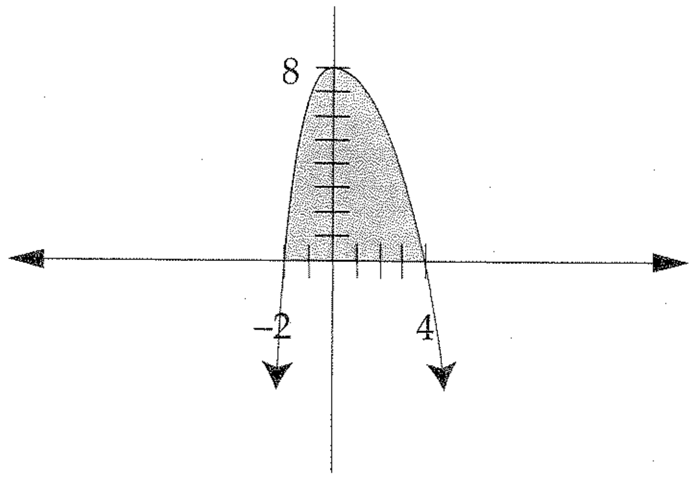
Therefore, the limit exists and $\lim_{x\to 4}\frac{x^2-16}{x-4} = 8$.

Now, let's check condition (3). In order for this condition to be fulfilled, k must equal 8.

The answer is (D).

Problem 8. Which of the following integrals correctly gives the area of the region consisting of all points above the x-axis and below the curve $y = 8 + 2x - x^2$?

The curve $y = 8 + 2x - x^2$ is an upside-down parabola and looks like this:



Notice that it crosses the x-axis at $x = -2$ and at $x = 4$.

The formula for the area of the region under the curve $f(x)$ and above the x-axis from $x = a$ to $x = b$ is: $\int_a^b f(x)\,dx$.

Thus, in order to find the area of the desired region, we need to evaluate the integral $\int_{-2}^{4} \left(8 + 2x - x^2\right) dx$.

The answer is (C).

Problem 9. If $f(x) = x^2 \cos 2x$, find $f'(x)$.

Here we need to use the product rule, which is:

If $f(x) = uv$, where u and v are both functions of x,

then $f'(x) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

Here, we get:

$$f'(x) = x^2(-2\sin 2x) + 2x(\cos 2x)$$

The answer is (D).

Problem 10. An equation of the line tangent to $y = 4x^3 - 7x^2$ at $x = 3$ is

If we want to find the equation of the tangent line, first we need to find the y-coordinate that corresponds to $x = 3$. It is:

$$y = 4(3)^3 - 7(3)^2 = 108 - 63 = 45.$$

Next, we need to find the derivative of the curve at $x = 3$. It is

$$\frac{dy}{dx} = 12x^2 - 14x \text{ and at } x = 3, \left.\frac{dy}{dx}\right|_{x=3} = 12(3)^2 - 14(3) = 66$$

Now we have the slope of the tangent line and a point that it goes through. We can use the point-slope formula for the equation of a line, $(y - y_1) = m(x - x_1)$, and Plug In what we have just found. We get:

$$(y - 45) = 66(x - 3)$$

The answer is (B).

Problem 11. $\int_0^{\frac{1}{2}} \frac{2}{\sqrt{1-x^2}}\,dx =$

This integral is of the form $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$, where $a = 1$.

Thus, we get:

$$\int_0^{\frac{1}{2}} \frac{2dx}{\sqrt{1-x^2}} = 2\sin^{-1}(x)\Big|_0^{\frac{1}{2}} = 2\left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)\right] = 2\left(\frac{\pi}{6} - 0\right) = \frac{\pi}{3}$$

The answer is (B).

Problem 12. Find a positive value c that satisfies the conclusion of the Mean Value Theorem for Derivatives for $f(x) = 3x^2 - 5x + 1$ on the interval $[2, 5]$.

The Mean Value Theorem for Derivatives says that, given a function $f(x)$ which is continuous and differentiable on $[a, b]$, then there exists some value c on (a, b) where

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Here, we have $\frac{f(b) - f(a)}{b - a} = \frac{f(5) - f(2)}{5 - 2} = \frac{51 - 3}{3} = 16$

and $f'(c) = 6c - 5$, so we simply set $6c - 5 = 16$

If we solve for c, we get: $c = \frac{7}{2}$

The answer is (E).

Problem 13. Given $f(x)=2x^2-7x-10$ find the absolute maximum of $f(x)$ on $[-1,3]$.

First, let's take the derivative and then set it equal to zero to determine any critical points of the function.

$$f'(x)=4x-7$$

$$4x-7=0$$

$$x=\frac{7}{4}$$

Now we can use the second derivative test to determine if this is a local minimum or maximum.

$$f''(x)=4$$

Because the second derivative is always positive, the function is concave up everywhere and thus $x=\frac{7}{4}$ must be a local minimum.

How, then, do we find the absolute maximum? Anytime we are given a function that is defined on an interval, the endpoints of the interval are also critical points. Thus, all that we have to do now is to plug the endpoints into the function and see which one gives us the bigger value. That will be the absolute maximum.

$$f(-1)=2(-1)^2-7(-1)-10=-1$$

$$f(3)=2(3)^2-7(3)-10=-13$$

Therefore, the absolute maximum of $f(x)$ on the interval $[-1,3]$ is -1.

The answer is (A).

Problem 14. Find $\frac{dy}{dx}$ if $x^3y+xy^3=-10$.

We need to use implicit differentiation to find $\frac{dy}{dx}$.

$$3x^2y+x^3\frac{dy}{dx}+y^3+3xy^2\frac{dy}{dx}=0$$

Now, in order to isolate $\frac{dy}{dx}$, we move all of the terms that do not contain $\frac{dy}{dx}$ to the right side of the equals sign:

$$x^3\frac{dy}{dx}+3xy^2\frac{dy}{dx}=-3x^2y-y^3$$

Factor out $\frac{dy}{dx}$:

$$\frac{dy}{dx}\left(x^3+3xy^2\right)=-3x^2y-y^3$$

And divide both sides by $\left(x^3+3xy^2\right)$ to isolate $\frac{dy}{dx}$:

$$\frac{dy}{dx}=-\frac{3x^2y+y^3}{x^3+3xy^2}$$

The answer is (D).

PROBLEM 15. If $f(x)=\sqrt{1+\sqrt{x}}$, find $f'(x)$.

First rewrite the equation using fractional powers instead of radical signs.

$$f(x)=\left(1+x^{\frac{1}{2}}\right)^{\frac{1}{2}}$$

Now take the derivative:

$$f'(x)=\frac{1}{2}\left(1+x^{\frac{1}{2}}\right)^{-\frac{1}{2}}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$

This can be rewritten as:

$$f'(x)=\frac{1}{4}\frac{1}{\sqrt{x}}\frac{1}{\sqrt{1+\sqrt{x}}}$$

The answer is (D).

PROBLEM 16. $\int 7xe^{3x^2}dx=$

We can use u-substitution to evaluate the integral.

Let $u=3x^2$ and $du=6xdx$. If we solve the second term for $x\,dx$, we get:

$$\frac{1}{6}du=xdx$$

Now we can rewrite the integral as:

$$\frac{7}{6}\int e^u\,du$$

Evaluate the integral to get:

$$\frac{7}{6}e^{u}+C$$

Now substitute back to get:

$$\frac{7}{6}e^{3x^2}+C$$

The answer is (C).

Problem 17. Find the equation of the tangent line to $9x^2+16y^2=52$ through $(2,-1)$.

First, we need to find $\frac{dy}{dx}$. It's simplest to find it implicitly:

$$18x+32y\frac{dy}{dx}=0$$

Now solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx}=-\frac{18x}{32y}=-\frac{9x}{16y}$$

Next, Plug In $x=2$ and $y=-1$ to get the slope of the tangent line at the point:

$$\frac{dy}{dx}=\frac{-18}{-16}=\frac{9}{8}$$

Now use the point-slope formula to find the equation of the tangent line:

$$(y+1)=\frac{9}{8}(x-2)$$

If we multiply through by 8, we get: $8y+8=9x-18$ or $9x-8y-26=0$.

The answer is (B).

Problem 18. A particle's position is given by $s=t^3-6t^2+9t$. What is its acceleration at time $t=4$?

Acceleration is the second derivative of position with restrict to time (Velocity is the first derivative).

The first derivative is: $v(t)=3t^2-12t+9$

The second derivative is: $a(t)=6t-12$

Now we simply Plug In $t=4$ and we get: $a(4)=24-12=12$

The answer is (E).

PROBLEM 19. If $f(x)=3^{\pi x}$, then $f'(x)=$

The derivative of an expression of the form a^u, where u is a function of x, is:

$$\frac{d}{dx}a^u = a^u \cdot \ln a \cdot \frac{du}{dx}$$

Here, we get:

$$\frac{d}{dx}3^{\pi x} = 3^{\pi x} \cdot \ln 3 \cdot \pi$$

The answer is (E).

PROBLEM 20. The average value of $f(x)=\frac{1}{x}$ from $x = 1$ to $x = e$ is

In order to find the average value, we use the Mean Value Theorem for Integrals, which says that the average value of $f(x)$ on the interval $[a,b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

Here, we have $\frac{1}{e-1}\int_1^e \frac{1}{x}\,dx$.

Evaluating the integral, we get: $\ln x\big|_1^e = \ln e - \ln 1 = 1$. Therefore, the answer is $\frac{1}{e-1}$.

The answer is (E).

PROBLEM 21. If $f(x)=\sin^2 x$, find $f'''(x)$.

We just use the chain rule three times.

$$f'(x) = 2\sin x \cos x = \sin 2x$$

$$f''(x) = 2\cos 2x$$

$$f'''(x) = -4\sin 2x$$

The answer is (D).

PROBLEM 22. Find the slope of the normal line to $y = x + \cos xy$ at $(0,1)$.

First, we need to find $\frac{dy}{dx}$ using implicit differentiation.

$$\frac{dy}{dx} = 1 - \left(x\frac{dy}{dx} + y\right)\sin xy$$

Rather than simplifying this, simply Plug In $(0,1)$ to find $\frac{dy}{dx}$.

We get: $\frac{dy}{dx} = 1$.

This means that the slope of the tangent line at $(0,1)$ is 1, so the slope of the normal line at $(0,1)$ is the negative reciprocal, which is -1.

The answer is (B).

Problem 23. $\int e^x \cdot e^{3x} dx =$

First, add the exponents to get: $\int e^{4x} dx$

Evaluating the integral, we get: $\frac{1}{4}e^{4x} + C$

The answer is (B).

Problem 24. $\lim_{x \to 0} \frac{\tan^3(2x)}{x^3} =$

We will need to use the fact that $\lim_{x \to 0} \frac{\sin x}{x} = 1$ to find the limit.

First, rewrite the limit as

$$\lim_{x \to 0} \frac{\sin^3(2x)}{x^3 \cos^3(2x)}$$

Next, break the fraction into:

$$\lim_{x \to 0}\left(\frac{\sin^3(2x)}{x^3} \frac{1}{\cos^3(2x)}\right)$$

Now, if we multiply the top and bottom of the first fraction by 8, we get:

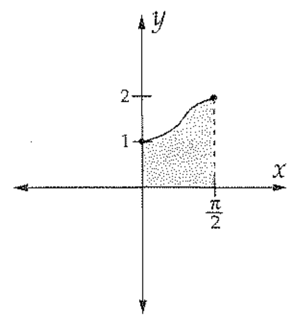
$$\lim_{x \to 0} \frac{8\sin^3(2x)}{(2x)^3} \frac{1}{\cos^3(2x)}$$

Now, we can take the limit, which gives us: $8 \cdot 1 \cdot 1 = 8$.

The answer is (D).

Problem 25. A solid is generated when the region in the first quadrant bounded by the graph of $y = 1+\sin^2 x$, the line $x = \frac{\pi}{2}$, the x-axis, and the y-axis is revolved about the x-axis. Its volume is found by evaluating which of the following integrals?

First, let's graph the curve.



We can find the volume by taking a vertical slice of the region. The formula for the volume of a solid of revolution around the x-axis, using a vertical slice bounded from above by the curve $f(x)$ and from below by $g(x)$, on the interval $[a,b]$, is:

$$\pi\int_a^b \left[f(x)^2 - g(x)^2\right]dx$$

Here, we get:

$$\pi\int_0^{\frac{\pi}{2}} \left(1+\sin^2 x\right)^2 dx$$

The answer is (D).

Problem 26. If $y = \left(\frac{x^3-2}{2x^5-1}\right)^4$, find $\frac{dy}{dx}$ at $x = 1$

We use the Chain Rule and the Quotient Rule.

$$\frac{dy}{dx} = 4\left(\frac{x^3-2}{2x^5-1}\right)^3\left[\frac{(2x^5-1)(3x^2)-(x^3-2)(10x^4)}{(2x^5-1)^2}\right]$$

If we Plug In 1 for x, we get:

$$\frac{dy}{dx} = 4(-1)^3\left[\frac{3+10}{1^2}\right] = -52$$

The answer is (A).

Problem 27. $\int x\sqrt{5-x}\,dx =$

We can evaluate this integral using u-substitution.

Let $u = 5 - x$ and $5 - u = x$. Then $-du = dx$.

Substituting, we get:

$$-\int (5-u)u^{\frac{1}{2}}\,du$$

The integral can be rewritten as:

$$-\int \left(5u^{\frac{1}{2}} - u^{\frac{3}{2}}\right) du$$

Evaluating the integral, we get:

$$-5\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + \frac{u^{\frac{5}{2}}}{\frac{5}{2}} + C$$

This can be simplified to:

$$-\frac{10}{3}u^{\frac{3}{2}} + \frac{2}{5}u^{\frac{5}{2}} + C$$

Finally, substituting back, we get:

$$-\frac{10}{3}(5-x)^{\frac{3}{2}} + \frac{2}{5}(5-x)^{\frac{5}{2}} + C$$

The answer is (E).

Problem 28. If $\frac{dy}{dx} = \frac{x^3+1}{y}$ and $y = 2$ when $x = 1$, then, when $x = 2$, $y =$

This is a differential equation that can be solved using separation of variables. Put all of the terms containing y on the left and all of the terms containing x on the right. We get:

$$y\,dy = \left(x^3 + 1\right)dx$$

Next, we integrate both sides:

$$\int y\,dy = \int \left(x^3 + 1\right)dx$$

Evaluating the integrals, we get:

$$\frac{y^2}{2}=\frac{x^4}{4}+x+C$$

Next, we Plug In $y=2$ and $x=1$ to solve for C. We get: $2=\frac{1}{4}+1+C$ and so $C=\frac{3}{4}$. This gives us:

$$\frac{y^2}{2}=\frac{x^4}{4}+x+\frac{3}{4}$$

Now, if we substitute $x=2$, we get:

$$\frac{y^2}{2}=4+2+\frac{3}{4}=\frac{27}{4}$$

Solving for y, we get:

$$y=\pm\sqrt{\frac{27}{2}}$$

The answer is (E).

Problem 29. The graph of $y=5x^4-x^5$ has an inflection point (or points) at

In order to find the inflection point(s) of a polynomial, we need to find the values of x where its second derivative is zero.

First, we find the second derivative.

$$\frac{dy}{dx}=20x^3-5x^4$$

$$\frac{d^2y}{dx^2}=60x^2-20x^3$$

Now, let's set the second derivative equal to zero and solve for x.

$$60x^2-20x^3=0$$

$$20x^2(3-x)=0$$

$$x=3$$

This is the point of inflection. $x=0$ is not a point of inflection because $\frac{d^2y}{dx^2}$ does not change sign there. If you are unsure that these are correct, graph the function with a calculator and look at the picture.

The answer is (B).

Problem 30. The average value of $f(x) = e^{4x^2}$ on the interval $\left[-\frac{1}{4}, \frac{1}{4}\right]$ is

In order to find the average value, we use the Mean Value Theorem for Integrals, which says that the average value of $f(x)$ on the interval $[a,b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

Here, we have

$$\frac{1}{\frac{1}{4}+\frac{1}{4}}\int_{-\frac{1}{4}}^{\frac{1}{4}} e^{4x^2}\,dx = 2\int_{-\frac{1}{4}}^{\frac{1}{4}} e^{4x^2}\,dx$$

You can't evaluate this integral using any of the techniques that you have studied so far, so use the calculator to evaluate the integral numerically.

Remember this: Any integral on the AP that contains an e^{x^2} term that is not multiplied by x, must be integrated using the calculator.

You should get approximately 1.09 . **The AP always expects you to round to three decimal places.**

The answer is (C).

Problem 31. $\int_0^1 \tan x\,dx =$

First, rewrite the integral as $\int_0^1 \frac{\sin x}{\cos x}\,dx$.

Now, we can use u-substitution to evaluate the integral. Let $u = \cos x$. Then $du = -\sin x$. We can also change the limits of integration. The lower limit becomes $\cos 0 = 1$ and the upper limit becomes $\cos 1$, which we leave alone. Now we perform the substitution and we get:

$$-\int_1^{\cos 1} \frac{du}{u}$$

Evaluating the integral, we get: $-\ln u\Big|_1^{\cos 1} = -\ln(\cos 1) + \ln 1 = -\ln(\cos 1)$. This log is also equal to $\ln(\sec 1)$.

The answer is (D).

PROBLEM 32. $\dfrac{d}{dx}\displaystyle\int_0^{x^2} \sin^2 t\, dt =$

The Second Fundamental Theorem of Calculus tells us how to find the derivative of an integral. It says that $\dfrac{d}{dx}\displaystyle\int_c^u f(t)\, dt = f(u)\dfrac{du}{dx}$, where c is a constant and u is a function of x.

Here we can use the theorem to get:

$$\frac{d}{dx}\int_0^{x^2} \sin^2 t\, dt = \left(\sin^2\left(x^2\right)\right)(2x) \text{ or } 2x\sin^2\left(x^2\right)$$

The answer is (B).

PROBLEM 33. Find the value(s) of $\dfrac{dy}{dx}$ of $x^2y + y^2 = 5$ at $y = 2$.

Here, we use implicit differentiation to find $\dfrac{dy}{dx}$:

$$2xy + x^2\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

Now we plug $y = 1$ into the original equation to find its corresponding x values.

$$x^2 + 1 = 5$$

$$x^2 = 4$$

$$x = \pm 2$$

Now Plug In the x and y values to find the value of $\dfrac{dy}{dx}$.

For $y = 1$ and $x = 2$, we get:

$$2(2)(1) + (2)^2\frac{dy}{dx} + 2(1)\frac{dy}{dx} = 0$$

Solving for $\dfrac{dy}{dx}$, we get:

$$4 + 6\frac{dy}{dx} = 0 \text{ and } \frac{dy}{dx} = -\frac{2}{3}$$

For $y = 1$ and $x = -2$, we get:

$$2(-2)(1) + (-2)^2\frac{dy}{dx} + 2(1)\frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$, we get:

$$-4+6\frac{dy}{dx}=0 \text{ and } \frac{dy}{dx}=\frac{2}{3}$$

The answer is (D).

Problem 34. The graph of $y=x^3-2x^2-5x+2$ has a local maximum at

First, let's find $\frac{dy}{dx}$:

$$\frac{dy}{dx}=3x^2-4x-5$$

Next, set the derivative equal to zero and solve for x.

$$3x^2-4x-5=0$$

Using the quadratic formula (or your calculator), we get:

$$x=\frac{4\pm\sqrt{16+60}}{6}\approx 2.120,-0.786$$

Let's use the second derivative test to determine which is the maximum. We take the second derivative and then Plug In the critical values that we found when we set the first derivative equal to zero. **If the sign of the second derivative at a critical value is positive, then the curve has a local minimum there. If the sign of the second derivative is negative, then the curve has a local maximum there.**

The second derivative is: $\frac{d^2y}{dx^2}=6x-4$. This is negative at $x=-0.786$, so the curve has a local maximum there. Now we plug $x=-0.786$ into the original equation to find the y-coordinate of the maximum. We get approximately 4.209. Therefore, the curve has a local maximum at $(-0.786, 4.209)$.

The answer is (D).

Problem 35. Approximate $\int_0^1 \sin^2 x\,dx$ using the trapezoid rule with $n=4$, to three decimal places.

The Trapezoid Rule enables us to approximate the area under a curve with a fair degree of accuracy. The rule says that the area between the x-axis and the curve $y=f(x)$, on the interval $[a,b]$, with n trapezoids, is:

$$\frac{1}{2}\frac{b-a}{n}\left[y_0+2y_1+2y_2+2y_3+\ldots+2y_{n-1}+y_n\right]$$

Using the rule here, with $n=4$, $a=0$, and $b=1$, we get:

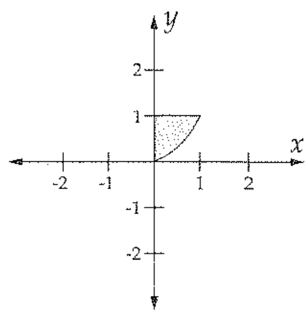
$$\frac{1}{2}\cdot\frac{1}{4}\left[\sin^2 0+2\sin^2\frac{1}{4}+2\sin^2\frac{1}{2}+2\sin^2\frac{3}{4}+\sin^2 1\right]$$

This is approximately 0.277.

The answer is (A).

PROBLEM 36. The volume generated by revolving about the x-axis the region above the curve $y=x^3$, below the line $y=1$, and between $x=0$ and $x=1$ is

First, make a quick sketch of the region.



We can find the volume by taking a vertical slice of the region. The formula for the volume of a solid of revolution around the x-axis, using a vertical slice bounded from above by the curve $f(x)$ and from below by $g(x)$, on the interval $[a,b]$, is:

$$\pi\int_a^b [f(x)]^2-[g(x)]^2\,dx$$

Here, we get:

$$\pi\int_0^1 (1)^2-(x^3)^2\ dx$$

Now we have to evaluate the integral. First, expand the integrand to get:

$$\pi\int_0^1 (1-x^6)dx$$

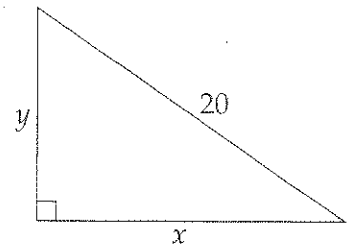
Next, integrate to get:

$$\pi\left(x-\frac{x^7}{7}\right)\Bigg|_0^1=\pi\left(1-\frac{1}{7}\right)=\frac{6\pi}{7}$$

The answer is (E).

Problem 37. A 20 foot ladder slides down a wall at 5 ft/sec. At what speed is the bottom sliding out when the top is 10 feet from the floor? (in ft/sec.)

First, let's make a sketch of the situation:



We are given that $\frac{dy}{dt} = -5$ (it's negative because the ladder is sliding down and its customary to make the upward direction positive), and we want to find $\frac{dx}{dt}$ when $y = 10$.

We can find a relationship between x and y using the Pythagorean Theorem. We get: $x^2 + y^2 = 400$.

Now, taking the derivative with respect to t, we get:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0, \text{ which can be simplified to } x\frac{dx}{dt} = -y\frac{dy}{dt}$$

Next, we need to find x when $y = 10$.

Using the Pythagorean Theorem,

$$x^2 + 10^2 = 400, \text{ so } x = \sqrt{300} \approx 17.321$$

Now, Plug Into the equation above to get:

$$17.321\frac{dx}{dt} = -10(-5) \text{ and } \frac{dx}{dt} \approx 2.887$$

The answer is (B).

PROBLEM 38. $\int \frac{\ln x}{3x}\,dx =$

We can evaluate the integral with u-substitution.

Let $u = \ln x$. Then $du = \frac{dx}{x}$.

Substituting, we get: $\frac{1}{3}\int u\,du$.

Now, we can evaluate the integral: $\frac{1}{3}\frac{u^2}{2} + C$.

Substituting back, we get: $\frac{\ln^2|x|}{6} + C$.

The answer is (D).

PROBLEM 39. Find two nonnegative numbers x and y whose sum is 100 and for which x^2y is a maximum.

Let's set $P = x^2y$. We want to maximize P, so we need to eliminate one of the variables. We are also given that $x + y = 100$, so we can solve this for y and substitute. $y = 100 - x$, so $P = x^2(100 - x) = 100x^2 - x^3$.

Now we can take the derivative.

$$\frac{dP}{dx} = 200x - 3x^2$$

Set the derivative equal to zero and solve for x.

$$200x - 3x^2 = 0$$

$$x(200 - 3x) = 0$$

$$x = 0 \text{ or } x = \frac{200}{3} \approx 66.667$$

Now we can use the second derivative to find the maximum. $\frac{d^2P}{dx^2} = 200 - 6x$.

If we Plug In $x = 66.667$, the second derivative is negative, so P is a maximum at $x = 66.667$. Solving for y, we get $y \approx 33.333$.

The answer is (E).

Problem 40. Find the distance traveled (to three decimal places) from $t = 1$ to $t = 5$ seconds, for a particle whose velocity is given by $v(t) = t + \ln t$.

The function $t + \ln t$ is always positive on the interval, so we can find the distance traveled by evaluating the integral

$$\int_1^5 (t + \ln t)\, dt$$

We can evaluate the integral numerically using the calculator.

You should get approximately 16.047 . **The AP always expects you to round to three decimal places.**

The answer is (C).

Problem 41. $\int \sin^5(2x)\cos(2x)\, dx =$

We can evaluate this integral using u-substitution.

Let $u = \sin(2x)$. Then $du = 2\cos(2x)dx$, which we can rewrite as $\frac{1}{2}du = \cos(2x)dx$.

Substituting into the integrand, we get:

$$\frac{1}{2}\int u^5 du$$

Evaluating the integral gives us:

$$\frac{1}{2}\frac{u^6}{6} + C = \frac{u^6}{12} + C$$

Substituting back, we get:

$$\frac{\sin^6(2x)}{12} + C$$

The answer is (A).

Problem 42. The volume of a cube is increasing at a rate proportional to its volume at any time t. If the volume is 8 ft^3 originally, and 12 ft^3 after 5 seconds, what is its volume at $t = 12$ seconds?

When we see a phrase where something is increasing at a rate "proportional to itself at any time t", this means that we set up the differential equation

$$\frac{dV}{dt} = kV$$

(or whatever the appropriate variable is)

We solve this differential equation using separation of variables.

First, move the V to the left side and the dt to the right side, to get:

$$\frac{dV}{V} = kdt$$

Now, integrate both sides:

$$\int \frac{dV}{V} = k\int dt$$

$$\ln V = kt + C$$

Next, it's traditional to put the equation in terms of V. We do this by exponentiating both sides to the base e. We get:

$$V = e^{kt+C}$$

Using the rules of exponents, we can rewrite this as:

$$V = e^{kt}e^{C}$$

Finally, because e^{C} is a constant, we can rewrite the equation as:

$$V = Ce^{kt}$$

Now, we use the initial condition that $V = 8$ at time $t = 0$ to solve for C.

$$8 = Ce^{0} = C(1) = C$$

This gives us

$$V = 8e^{kt}$$

Next, we use the condition that $V = 12$ at time $t = 5$ to solve for k.

$$12 = 8e^{5k}$$

$$\frac{3}{2} = e^{5k}$$

$$\ln\frac{3}{2} = 5k$$

$$k = \frac{1}{5}\ln\frac{3}{2}$$

This gives us:

$$V = 8e^{\left(\frac{1}{5}\ln\frac{3}{2}\right)t}$$

Finally, we Plug In $t = 12$ and solve for V:

$$V = 8e^{\left(\frac{1}{5}\ln\frac{3}{2}\right)(12)} \approx 21.169$$

The answer is (A).

PROBLEM 43. If $f(x) = \left(1 + \frac{x}{20}\right)^5$, find $f''(40)$.

The first derivative is:

$$f'(x) = 5\left(1 + \frac{x}{20}\right)^4\left(\frac{1}{20}\right) = \frac{1}{4}\left(1 + \frac{x}{20}\right)^4$$

The second derivative is:

$$f''(x) = \frac{1}{4} \cdot 4\left(1 + \frac{x}{20}\right)^3\left(\frac{1}{20}\right) = \frac{1}{20}\left(1 + \frac{x}{20}\right)^3$$

Evaluating this at $x = 40$, we get:

$$f''(x) = \frac{1}{20}\left(1 + \frac{40}{20}\right)^3 = \frac{27}{20} = 1.350$$

The answer is (B).

PROBLEM 44. A particle's height at a time $t \geq 0$ is given by $h(t) = 100t - 16t^2$. What is its maximum height?

First, let's take the derivative: $h'(t) = 100 - 32t$

Now, we set it equal to zero and solve for t: $100 - 32t = 0$

$$t = \frac{100}{32}$$

Now, to solve for the maximum height, we simply plug $t = \frac{100}{32}$ back into the original equation for height:

$$h\left(\frac{100}{32}\right) = 100\left(\frac{100}{32}\right) - 16\left(\frac{100}{32}\right)^2 = 156.250$$

By the way, we know that this is a maximum not a minimum because the second derivative is -32, which means that the critical value will give us a maximum not a minimum.

The answer is (B).

Problem 45. If $f(x)$ is continuous and differentiable and

$$f(x)=\begin{cases} ax^4+5x\,;\,x\le 2 \\ bx^2-3x\,;\,x>2 \end{cases},\text{ then } b=$$

In order to solve this for b, we need $f(x)$ to be differentiable at $x=2$.

If we plug $x=2$ into both pieces of this piecewise function, we get:

$$f(x)=\begin{cases} 16a+10;\, x\le 2 \\ 4b-6; \quad x>2 \end{cases},$$

so we need $16a+10=4b-6$.

Now, if we take the derivative of both pieces of this function and Plug In $x=2$ we get:

$$f'(x)=\begin{cases} 32a+5;\, x\le 2 \\ 4b-3; \quad x>2 \end{cases}, \text{ so we need } 32a+5=4b-3$$

Solving the simultaneous equations, we get $a=\frac{1}{2}$ and $b=6$.

The answer is (D).

ANSWERS AND EXPLANATIONS TO SECTION II

PROBLEM 1. Consider the curve defined by $y = x^4 + 4x^3$.

(a) If we want to find the equation of the tangent line, first we need to find the y-coordinate that corresponds to $x = -1$.

It is:

$$y = (-1)^4 + 4(-1)^3 = 1 - 4 = -3$$

Next, we need to find the derivative of the curve at $x = -1$.

It is $\frac{dy}{dx} = 4x^3 + 12x^2$ and, at $x = -1$, $\left.\frac{dy}{dx}\right|_{x=-1} = 4(-1)^3 + 12(-1)^2 = 8$.

Now we have the slope of the tangent line and a point that it goes through. We can use the point-slope formula for the equation of a line, $(y - y_1) = m(x - x_1)$, and Plug In what we have just found.

We get:

$$(y + 3) = 8(x + 1)\text{, which can be rewritten as } y = 8x + 5$$

(b) First, we set the derivative equal to zero and solve for x.

$$\frac{dy}{dx} = 4x^3 + 12x^2 = 0$$

$$4x^2(x + 3) = 0$$

$$x = 0 \text{ or } x = -3$$

Now, we can use the second derivative test to determine whether a critical value is the x-coordinate of a minimum or a maximum. The second derivative test is the following:

If c is a critical point, then:

c is the x-coordinate of a maximum if $f''(c) < 0$ and

c is the x-coordinate of a minimum if $f''(c) > 0$.

By the way, **c is the x-coordinate of a point of inflection if $f''(c) = 0$.**

So now we need to find the second derivative.

$$\frac{d^2y}{dx^2} = 12x^2 + 24x$$

If we Plug In $x = -3$, we get

$$\frac{d^2y}{dx^2} = 12(-3)^2 + 24(-3) = 36,$$

so the curve has a minimum at $x = -3$. Finally, to get the y-coordinate of the minimum, we plug $x = -3$ into the original equation and we get

$$y = (-3)^4 + 4(-3)^3 = 81 - 108 = -27$$

Thus, the curve has an absolute minimum at $(-3, -27)$.

(c) If we Plug In $x = 0$ we get

$$\frac{d^2y}{dx^2} = 12(0)^2 + 24(0) = 0,$$

so the curve has a point of inflection at $x = 0$. To get the y-coordinate of the point of inflection, we plug $x = 0$ into the original equation and we get

$$y = (0)^4 + 4(0)^3 = 0$$

Thus, the curve has a point of inflection at $(0, 0)$.

Problem 2. The temperature on New Year's Day in Hinterland was given by $T(H) = -A - B\cos\left(\frac{\pi H}{12}\right)$, where T is the temperature in degrees Fahrenheit and H is the number of hours from midnight $0 \le T < 24$.

(a) Simply Plug In the temperature, -15, for T and the time, midnight ($T = 0$), for H into the equation. We get: $-15 = -A - B\cos 0$, which simplifies to $-15 = -A - B$.

Now plug the temperature, 5, for T and the time, Noon ($H = 12$), for H into the equation. We get: $5 = -A - B\cos(\pi)$, which simplifies to $5 = -A + B$.

Now we can solve the pair of simultaneous equations for A and B, and we get $A = 5^\circ F$ and $B = 10^\circ F$.

(b) In order to find the average value, we use the Mean Value Theorem for Integrals, which says that the average value of $f(x)$ on the interval $[a,b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

Here, we have $\frac{1}{10-0}\int_0^{10}\left(-5-10\cos\left(\frac{\pi H}{12}\right)\right)dH$

Evaluating the integral, we get:

$$\frac{1}{10}\left[\left(-5H-\frac{120}{\pi}\sin\left(\frac{\pi H}{12}\right)\right)\right]_0^{10}=\frac{1}{10}\left[-50-\frac{120}{\pi}\sin\left(\frac{5\pi}{6}\right)\right]=\frac{1}{10}\left(-50-\frac{60}{\pi}\right)\approx -6.910^\circ F$$

(c) The Trapezoid Rule enables us to approximate the area under a curve with a fair degree of accuracy. The rule says that the area between the x-axis and the curve $y=f(x)$, on the interval $[a,b]$, with n trapezoids, is:

$$\frac{1}{2}\frac{b-a}{n}\left[y_0+2y_1+2y_2+2y_3+\ldots+2y_{n-1}+y_n\right]$$

Using the rule here, with $n=4$, $a=6$, and $b=8$, we get:

$$\frac{1}{2}\cdot\frac{1}{2}\left[\left(-5-10\cos\frac{6\pi}{12}\right)+2\left(-5-10\cos\frac{13\pi}{24}\right)+2\left(-5-10\cos\frac{7\pi}{12}\right)+2\left(-5-10\cos\frac{15\pi}{24}\right)+\left(-5-10\cos\frac{8\pi}{12}\right)\right]$$

This is approximately $-4.890^\circ F$.

(d) We simply take the derivative with respect to H.

$$\frac{dT}{dH}=-10\left(\frac{\pi}{12}\right)\left(-\sin\frac{\pi H}{12}\right)=\frac{5\pi}{6}\sin\frac{\pi H}{12}$$

Problem 3. Sea grass grows on a lake. The rate of growth of the grass is $\frac{dG}{dt}=kG$, where k is a constant.

(a) We solve this differential equation using separation of variables.

First, move the G to the left side and the dt to the right side, to get: $\frac{dG}{G}=kdt$.

Now, integrate both sides:

$$\int\frac{dG}{G}=k\int dt$$

$$\ln G = kt + C$$

Next, solve for G by exponentiating both sides to the base e. We get: $G = e^{kt+C}$

Using the rules of exponents, we can rewrite this as: $G = e^{kt}e^{C}$. Finally, because e^{C} is a constant, we can rewrite the equation as: $G = Ce^{kt}$.

Now, we use the initial condition that $G = 100$ at time $t = 0$ to solve for C.

$$100 = Ce^{0} = C(1) = C$$

This gives us $G = 100e^{kt}$.

Next, we use the condition that $G = 120$ at time $t = 1$ to solve for k.

$$120 = 100e^{k}$$

$$1.2 = e^{k}$$

$$\ln 1.2 = k \approx 0.1823$$

This gives us: $G = 100e^{0.1823t}$

(b) All we need to do is set G equal to 300 and solve for t.

$$300 = 100e^{0.1823t}$$

$$3 = e^{0.1823t}$$

$$\ln 3 = 0.1823t$$

$$t \approx 6.026 \text{ years}$$

(c) Now we have to account for the fish's consumption of the sea grass. So we have to evaluate the differential equation $\frac{dG}{dt} = kG - 80$.

First, separate the variables, to get:

$$\frac{dG}{kG - 80} = dt$$

Now, integrate both sides:

$$\int \frac{dG}{kG - 80} = \int dt \text{ or } \int \frac{dG}{G - \frac{80}{k}} = k\int dt$$

$$\ln\left(G - \frac{80}{k}\right) = kt + C$$

Next, exponentiate both sides to the base e. We get:

$$G - \frac{80}{k} = Ce^{kt}$$

Solving for G, we get:

$$G = \left(G_0 - \frac{80}{k}\right)e^{kt} + \frac{80}{k}$$

Now, set $G = 0$. We get:

$$0 = \left(G_0 - \frac{80}{k}\right)e^{kt} + \frac{80}{k}$$

Now, set $G_0 = 300$ and rearrange:

$$kt = \ln \frac{\frac{-80}{k}}{\left(300 - \frac{80}{k}\right)} = \ln \frac{80}{80 - 300k}$$

Take the log of both sides:

$$kt = \ln\left(\frac{80}{80 - 300k}\right)$$

$$\text{and } t = \frac{1}{k}\ln\left(\frac{80}{80 - 300k}\right)$$

Now, we Plug In the value for k that we got in part (a) above and we get $t \approx 6.313$ years.

Problem 4. Water is being poured into a hemispherical bowl of radius 6 inches at the rate of 4 in^3/sec.

(a) First, rewrite the equation as

$$V = \pi R h^2 - \frac{\pi}{3}h^3$$

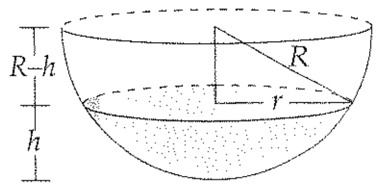
Now take the derivative of the equation with respect to t:

$$\frac{dV}{dt} = 2\pi R h \frac{dh}{dt} - \pi h^2 \frac{dh}{dt}$$

If we Plug In $\frac{dV}{dt}=4$, $R=6$, and $h=2$, we get:

$$4=20\pi\frac{dh}{dt} \text{ or } \frac{dh}{dt}=\frac{4}{20\pi}=\frac{1}{5\pi}\approx 0.064 \text{ in/sec}$$

(b) Notice that we can construct a right triangle using the radius of the sphere and the radius of the surface of the water.



Notice that the distance from the center of the sphere to the surface of the water is $R-h$. Now, we can use the Pythagorean Theorem to find r:

$$R^2=(R-h)^2+r^2$$

We can rearrange this to get:

$$r=\sqrt{R^2-(R-h)^2}=\sqrt{2Rh-h^2}$$

Because $R=6$, we get:

$$r=\sqrt{12h-h^2}$$

(c) The area of the surface of the water is $A=\pi r^2$, where $r=\sqrt{12h-h^2}$. Thus, $A=\pi\left(12h-h^2\right)$.

Taking the derivative of the equation with respect to t, we get:

$$\frac{dA}{dt}=\pi\left(12\frac{dh}{dt}-2h\frac{dh}{dt}\right)$$

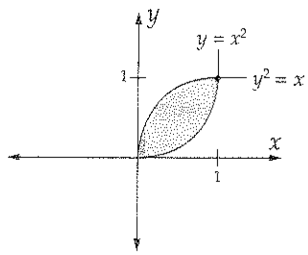
We found in part (a) above that

$$\frac{dh}{dt}=\frac{1}{5\pi}, \text{ so } \frac{dA}{dt}=\pi\left(\frac{12}{5\pi}-\frac{4}{5\pi}\right)=\frac{8}{5} \text{ in}^2\text{/sec.}$$

PROBLEM 5. Let R be the region in the first quadrant bounded by $y^2 = x$ and $x^2 = y$.

(a) Find the area of region R.

First, let's sketch the region.



In order to find the area, we "slice" the region vertically and add up all of the slices. Now, we use the formula for the area of the region between $y = f(x)$ and $y = g(x)$, from $x = a$ to $x = b$,

$$\int_a^b [f(x) - g(x)]\,dx$$

We need to rewrite the equation $y^2 = x$ as $y = \sqrt{x}$ so that we can integrate with respect to x. Our integral for the area is:

$$\int_0^1 \left(\sqrt{x} - x^2\right) dx$$

Evaluating the integral, we get:

$$\left(\frac{2x^{\frac{3}{2}}}{3} - \frac{x^3}{3}\right)\Bigg|_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

(b) In order to find the volume of a region between $y = f(x)$ and $y = g(x)$, from $x = a$ to $x = b$, when it is revolved around the x-axis, we use the formula

$$\pi \int_a^b \left[f(x)^2 - g(x)^2 \right] dx$$

Here, our integral for the area is:

$$\pi \int_0^1 \left(x - x^4\right) dx$$

Evaluating the integral, we get:

$$\pi\left(\frac{x^2}{2}-\frac{x^5}{5}\right)\Bigg|_0^1=\pi\left(\frac{1}{2}-\frac{1}{5}\right)=\frac{3\pi}{10}$$

(c) Whenever we want to find the volume of a solid, formed by the region between $y=f(x)$ and $y=g(x)$,with a known cross-section, from $x=a$ to $x=b$, when it is revolved around the x-axis, we use the formula

$$\int_a^b A(x)\,dx$$

where $A(x)$ is the area of the cross section. We find the area of the cross-section by using the vertical slice formed by $f(x)-g(x)$, and then plugging it into the appropriate area formula. In the case of a circle, $f(x)-g(x)$ gives us the length of the diameter and we use the formula

$$A(x)=\frac{\pi(diameter)^2}{4}$$

This gives us the integral:

$$\int_0^1\frac{\pi}{4}\left(\sqrt{x}-x^2\right)^2 dx$$

Expand the integrand:

$$\int_0^1\frac{\pi}{4}\left(\sqrt{x}-x^2\right)^2 dx=\frac{\pi}{4}\int_0^1\left(x-2x^{\frac{5}{2}}+x^4\right)dx$$

Evaluate the integral.

$$\frac{\pi}{4}\int_0^1\left(x-2x^{\frac{5}{2}}+x^4\right)dx=\frac{\pi}{4}\left(\frac{x^2}{2}-\frac{4x^{\frac{7}{2}}}{7}+\frac{x^5}{5}\right)\Bigg|_0^1=\frac{\pi}{4}\left(\frac{1}{2}-\frac{4}{7}+\frac{1}{5}\right)=\frac{9\pi}{280}$$

Problem 6. An object moves with velocity $v(t)=t^2-8t+7$.

(a) The velocity of an object is the derivative of its position with respect to time. Thus, if we want to find the position, we take the integral of velocity with respect to time.

$$s(t)=\int\left(t^2-8t+7\right)dt=\frac{t^3}{3}-\frac{8t^2}{2}+7t+C=\frac{t^3}{3}-4t^2+7t+C$$

(b) If we want to find when the particle is changing direction, we need to find where the velocity of the particle is zero.

$$v(t)=t^2-8t+7=(t-1)(t-7)=0$$

Thus, at $t=1$ or $t=7$, the particle could be changing direction. To make sure, we need to check that the acceleration of the particle is not zero at those times. The acceleration of a particle is the derivative of the velocity with respect to time.

$$a(t)=2t-8$$

At $t=1$,

$$a(1)=2-8=-6,$$

which is not zero, so the particle is changing direction at $t=1$.

At $t=7$,

$$a(7)=14-8=6,$$

which is not zero, so the particle is changing direction at $t=7$.

(c) If we want to find the total distance that a particle travels from time a to time b, we need to evaluate

$$\int_a^b |v(t)|\,dt$$

This means that, over an interval where the particle's velocity is negative, we multiply the integral by -1. So, we need to find where the velocity is negative and where it is positive.

We know that the velocity is zero at $t=1$ and at $t=7$.

We can find that the velocity is positive when $t<1$ and when $t>7$, and the velocity is negative when $1<t<7$.

Thus, the distance that the particle travels from $t = 0$ to $t = 4$ is

$$\int_0^1 \left(t^2 - 8t + 7\right) dt - \int_1^4 \left(t^2 - 8t + 7\right) dt$$

Evaluating the integrals, we get:

$$\left(\frac{t^3}{3} - 4t^2 + 7t\right)\Bigg|_0^1 - \left(\frac{t^3}{3} - 4t^2 + 7t\right)\Bigg|_1^4 = \left(\frac{10}{3} - 0\right) - \left(-\frac{44}{3} - \frac{10}{3}\right) = \frac{64}{3}$$

ANSWER KEY TO SECTION 1

1.	B	11.	B	21.	D	31.	D	41.	A
2.	D	12.	E	22.	B	32.	B	42.	A
3.	B	13.	A	23.	B	33.	D	43.	B
4.	E	14.	D	24.	D	34.	D	44.	B
5.	A	15.	D	25.	D	35.	A	45.	D
6.	A	16.	C	26.	A	36.	E		
7.	D	17.	B	27.	E	37.	B		
8.	C	18.	E	28.	E	38.	D		
9.	D	19.	E	29.	B	39.	E		
10.	B	20.	E	30.	C	40.	C		